Maths for Computing
Assignment 4 Solutions

1. *(5 marks)* Let $n \geq 1$. Prove that if we select any $n + 1$ integers from $[2n]$, then there exists two integers, say a and b, such that $a \% b = 0$.

Solution: Let x be any number in $[2n]$. Then x can be written in the form of $2^k y$, where $k \geq 0$ and y is an odd number. Now, we can pick any $n + 1$ numbers from $[2n]$ and these will be our pigeons. Create n pigeonholes corresponding to each odd number in $[2n]$. Put a pigeon, say $x = 2^k y$, to the pigeonhole y. From pigeonhole principle, we can say that there will be at least one pigeonhole, say y, that contains two pigeons, say $a = 2^{k_1} y$ and $b = 2^{k_2} y$, such that $k_1 > k_2$. Clearly, a is divisible by b. Hence, $a \% b = 0$.

2. *(3 marks)* How many ways are there to select an 11-member soccer team and a 5-member basketball team from a class of 30 students if

a) nobody can be on two teams.

Solution: We can first pick soccer team in $\binom{30}{11}$ ways and then from the rest of the 19 players we can pick the basketball team in $\binom{19}{5}$ ways because no student from the soccer team must be there in the basketball team. So the total number of ways teams' selection can be done is $\binom{30}{11}\binom{19}{5}$.

b) any number of students can be on both the teams.

Solution: Since there are no restrictions we can first choose the soccer team in $\binom{30}{11}$ ways and then the basketball team in $\binom{30}{5}$ ways. Therefore, the total number of ways will be $\binom{30}{11}\binom{30}{5}$.

c) at most one student can be on both the teams.

Solution: First pick the student that is present in both the teams. This can be done in 30 ways. Then, from the rest of the 29 players we can pick the other 10 members of the soccer team in $\binom{29}{10}$ ways and then from the rest of the 19 players, we can pick other 4 members of basketball team in $\binom{19}{4}$ ways. So the total number of ways teams can be selected with exactly one common player is $30\binom{29}{10}\binom{19}{4}$. Add $\binom{30}{11}\binom{30}{5}$ to this number as it is the number of ways teams can be selected with no common player.

3. *(5 marks)* Give a combinatorial proof of the following equation, i.e., prove that both sides are counting the same thing, where $1 \leq k \leq n$ and k, n are integers. (Modifications on any side is not allowed)

$$1.\binom{n-1}{k-1} + 2.\binom{n-2}{k-1} + \ldots + (n-k+1)\,.\binom{k-1}{k-1} = \binom{n+1}{k+1}$$

Solution: The RHS is the number of $(k+1)$-element subsets of $[n+1]$.
The LHS counts the same thing but with respect to the position of the 2nd smallest element of the $(k+1)$-element subset. The number of ways to pick $(k+1)$-element subsets such that i is the 2nd smallest element is $(i-1)\,.\binom{n-i+1}{k-1}$ as we can pick the smallest element in $(i-1)$ ways and then the rest of the $k-1$ elements from the remaining $n+1-i\ \ (=n-i+1)$ elements. Finally, we sum $(i-1)\,.\binom{n-i+1}{k-1}$ over all possible values of i, which is $i = 2$ to $n-k+2$.

4. *(5 marks)* Suppose $X = [12]$ and $Y = [8]$. How many functions f from X to Y satisfy the property that $|f(X)| = 5$? How many satisfy the property that $|f(X)| \leq 5$?
Solutions:
The first question is asking to find the number of functions from X to Y such that size of the range is 5, i.e., $|f(X)| = 5$. We can first pick 5 elements from Y in $\binom{8}{5}$ ways and then for every set 5 elements the number of surjections is $5!S(12,5)$. Therefore, the final answer is $\binom{8}{5}5!S(12,5)$.

Similarly, for the second question the answer is $\binom{8}{5}5!S(12,5) + \binom{8}{4}4!S(12,4) + \binom{8}{3}$ $3!S(12,3) + \binom{8}{2}2!S(12,2) + \binom{8}{1}1!S(12,1)$.

Note that $\binom{8}{5}5^{12}$, i.e., picking 5 elements and then assigning them to 12 elements without any restriction, is not the right answer as it is overcounting. (Think about why the function that maps every element to 1 will be counted more than once in this answer.)

5. *(5 marks)* Prove that $p_3(n) = |X|$, where X is the set of 3 length partitions of $2n$, where every element of the partition is at most $n - 1$.
Solution: Let Y be the set of all partitions of n into 3 parts. The problem is asking to prove that $|Y| = |X|$.

We can do this by giving a bijection f between Y and X,

$$f((x_1, x_2, x_3)) = (n - x_3, n - x_2, n - x_1)$$

Now, we prove three things:
f's range is X:
For any partition of n, say (x_1, x_2, x_3), $(n - x_3, n - x_2, n - x_1)$ will be a 3 length partition of $2n$, where every element of the partition is at most $n - 1$ because
(i) $n - x_3 + n - x_2 + n - x_1 = 3n - (x_1 + x_2 + x_3) = 2n$,
(ii) $n - x_3 \geq n - x_2 \geq n - x_1$ as $x_1 \geq x_2 \geq x_3$, and
(iii) $n - x_i \leq n - 1$ because $x_i \geq 1$.

f is one to one:
Let (x_1, x_2, x_3) and (y_1, y_2, y_3) be two distinct 3 length partitions of n. Then, for some i $x_i \neq y_i$. Therefore, $(n - x_3, n - x_2, n - x_1) \neq (n - y_3, n - y_2, n - y_1)$ as for the same i, $n - x_i \neq n - y_i$.

f is onto:
Let (w_1, w_2, w_3) be some 3 length partition of $2n$, where each part is at most $n - 1$. Then, $f((x_1, x_2, x_3)) = (w_1, w_2, w_3)$, for $(x_1, x_2, x_3) = (n - w_3, n - w_2, n - w_1)$.

$(x_1, x_2, x_3) \in X$ because
(i) $n - w_3 + n - w_2 + n - w_1 = 3n - (w_1 + w_2 + w_3) = 3n - 2n = n$
(ii) $x_1 \leq x_2 \leq x_3$ as $w_1 \leq w_2 \leq w_3 \implies -w_1 \geq -w_2 \geq -w_3 \implies n - w_1 \geq n - w_2 \geq n - w_3$
(iii) $x_i \geq 1$ as $w_j \leq n - 1$

6. (5 marks) Prove that if $n \geq 2$, then $n! < S(2n, n) < (2n)!$.

Solution:

Upper Bound: Let $B(n)$ denote the number of all set partitions of $[n]$. In tutorial 9, we proved that $B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i)$. We will use this to prove that $B(n) < n!$, for $n \geq 3$, using strong induction.

Basis Step: $B(3) = 5 < 3!$ (Notice that $B(i) = i!$, for $i \in \{0,1,2\}$)

Inductive Step: Assuming $B(k) < k!$ for $k \geq 3$ and $B(k) = k!$ for $k \in \{0,1,2\}$, we will prove that $B(k+1) < (k+1)!$ for $k \geq 4$.

$$\begin{aligned} B(k+1) &= \sum_{i=0}^{k} \binom{k}{i} B(i) \\ &< \sum_{i=0}^{k} \binom{k}{i} i! \\ &< k! \sum_{i=0}^{k} \frac{1}{i!} \\ &< k!(k+1) \\ &< (k+1)! \end{aligned}$$

Now, $S(2n, n) < B(2n) < (2n)!$, for $n \geq 2$.

Lower Bound: Form a partition of $2n$ into n non-empty subsets by pairing every element of $\{1,2,\ldots,n\}$ with an element from $\{n+1, n+2, \ldots, 2n\}$. This can be done in $n!$ ways. Clearly, there are more partitions of $[2n]$ into n non-empty subsets, for $n \geq 2$, such as partitions where some subsets are not of size 2. Hence, $n! < S(2n, n)$.